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A REVIEW OF MAXIMUM-ENTROPY SPECTRAL ANALYSIS

by

R.W. Herring

DEPARTMENT OF COMMUNICATIONS
MINISTÈRE DES COMMUNICATIONS

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A REVIEW OF MAXIMUM-ENTROPY SPECTRAL ANALYSIS

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10 R.W. Herring

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ABSTRACT

The maximum-entropy method (MEM) of spectral analysis is a technique for estimating the power spectra of time-series data. The purpose of this note is to survey the literature of the MEM and to provide some insight into its derivation and usage. Fortran subroutines for computing the MEM power spectra of complex-valued data are included.

1. INTRODUCTION

The maximum-entropy method (MEM) of spectral analysis is a technique for estimating the power spectra of time-series data. When the "true" spectrum of the process being investigated consists of discrete lines separated by at least the reciprocal of the length of the data record,* the MEM yields a spectral estimate consisting of sharp, narrow spikes. This is in marked contrast to the performance of the classical Blackman and Tukey (1959) and periodogram techniques. These latter techniques estimate spectral lines as broad, smooth peaks which tend to merge if the lines are closely spaced in frequency. The discontinuous nature of the spectrum is thus masked.

---

* *The classical Rayleigh criterion.*

The purpose of this report is to survey the literature of the MEM (Section 2) and to provide some insight into its derivation and usage (Sections 3 and 4). Section 3 includes a derivation of an MEM algorithm suitable for analyzing complex-valued data records. The algorithm is not new (cf. Smylie et al., 1973), but the present derivation follows the more lucid approach taken by Andersen (1974) for real-valued data. Section 3 also includes a brief description of the criterion derived by Akaike (1969a,b; 1970) for selecting the most appropriate of the many possible choices of the MEM spectrum which can be derived from a particular set of data. Section 4 refers to listings of Fortran subroutines for computing the MEM spectra of complex-valued data. These listings are included as appendices.

MEM spectral analysis of complex-valued data is applicable to data derived from such systems as sampled-aperture antenna arrays, where the calculated spectra may be interpreted in terms of signal strength as a function of angle-of-arrival. The method should also be useful for the frequency analysis of tone-encoded data. In such applications the use of complex-valued data allows the unambiguous distinction between positive and negative frequencies, which often is necessary.

2. REVIEW

The two classical methods of spectral analysis are the periodogram technique and the Blackman and Tukey (1959) power spectrum technique. In using the periodogram technique, the squared magnitude of the discrete Fourier transform (DFT) of the data is computed. Nowadays the DFT is usually computed by means of the fast Fourier transform (FFT) algorithm. The amplitude (unsquared) spectrum is often smoothed (usually by the well-known technique of "windowing" the data before transformation) in order to reduce the excessive spreading or "leakage" of spectral lines caused by the finite length of the data record. Often the computed power (squared-magnitude) spectrum is also smoothed to improve its statistical reliability. Both types of smoothing reduce spectral resolution.

The Blackman and Tukey technique consists of two stages. In the first stage, the autocorrelation function of the data is estimated by extending a finite-length data record with zeros and then computing the autocorrelation function of this extended data record. Usually lag distances of not more than 10% of the original record length are considered. In the second stage of the procedure the DFT of the estimated autocorrelation function is computed to obtain a smoothed power-spectrum estimate. This technique was developed before the FFT algorithm was known, when the direct computation of DFTs was very laborious. It was intended for use in analyzing noise-like data with smoothly varying spectra (i.e., having no spectral lines), and it avoided the costly and frustrating necessity of first computing a spectrum in far greater detail than was required, only to subsequently smooth it in order to obtain the desired statistical stability.

The maximum-entropy method of spectral analysis was first proposed by Burg (1967, 1968) as a technique for improving spectral resolution in the sense of reducing the spectral smearing caused by the finite length of the data record. In the classical spectral analysis techniques described above,

the estimated autocorrelation function is tacitly extended either by zeros or by a cyclic repetition of itself. Burg proposed that a more appropriate extension of the autocorrelation function could be based on that extrapolation which gives rise to the statistically most random (i.e., maximum-entropy) power spectrum estimate consistent with the observed data. Burg (1968) showed how to calculate such a maximum-entropy spectrum from a real sampled data record. This technique and its extension to include complex-valued data is presented in Section 3 which follows.

Van de Bos (1971) demonstrated that the Burg maximum-entropy equations were mathematically the same as those encountered in the least-squares fitting of an all-pole spectral model to time-series data. He also pointed out that such a model, in which the data are assumed to have been generated by a set of damped resonators, is invalid for many kinds of data. This restriction must be taken into account whenever the use of the maximum-entropy method is proposed. In many practical cases*, however, the data can be considered as the sum of one or more truncated series of complex exponentials plus noise, and the all-pole model is appropriate.

Lacoss (1971) did a comprehensive analysis of both the MEM and the analytically somewhat similar maximum-likelihood method (MLM) of spectral analysis. The major result of this work was to obtain the shape and magnitude of the expected spectral-response function for the case where a small number of exactly known samples of the autocorrelation functions of real sinusoidal signals plus uncorrelated additive noise have been given. The crucial problem of estimating the autocorrelation function was not considered. It was also shown that if the autocorrelation function were known exactly, sinusoids with frequencies separated by amounts greater than the reciprocal of the length of sampled autocorrelation function would be sharply resolved. This is in marked contrast to the blurred, broad spectral peaks which occur in this case for the classical Blackman-Tukey and periodogram methods. It is recommended that the prospective user of the MEM examine the results of Lacoss' work closely, particularly in regard to the shape and magnitude of the expected spectral peaks.

Ulrych (1972) and Ulrych et al. (1973) devoted considerable effort to the problem of using the analysis techniques outlined by Burg (1968) to analyze short, real-data records containing one or more sinusoidal components. Their results included a comparison of the Fourier periodogram and maximum entropy spectra of the same records, and showed a dramatic sharpening of the spectral peaks estimated by the maximum entropy method relative to those estimated by the Fourier method.

Smylie et al. (1973), in a very complete and rigorous exposition, generalized Burg's analysis technique to include the case of complex sampled data. The extended technique was then used to analyze irregularities in the earth's rotation. An incomplete extension to complex data, based on the elegant recursive formulas derived by Andersen (1974), was published by Haykin and Kesler (1976), who seemed unaware of the earlier and more comprehensive work of Smylie et al..

---

* *e.g., short records of tone-encoded data; plane-wave data received by a linear sampled-aperture array of sensors.*

The most comprehensive review to date of the subject of MEM and MLM spectral-analysis techniques is that of Ulrych and Bishop (1975), which showed the applicability of the criterion derived by Akaike (1971) for determining from the data the optimal length of the spectral estimator (see Section 3).

Another excellent review of the subject was recently published by Kaveh and Cooper (1976). The relation between the MEM and the very similar autoregressive method is clearly brought out (cf. van den Bos, 1971) and a comparison is made between these methods and the classical Blackman and Tukey (1959) approach.

The problem of estimating confidence intervals for MEM spectral estimates has been considered by Baggeroer (1976), who included the case of estimating the confidence limits for spectral estimates based on complex data. This highly mathematical paper included a comparison of the results of Akaike (1969b).

Finally, the problem of estimating maximum-entropy spectra when a set of measured autocorrelation functions of unequal statistical weight are available has been investigated by Newman (1977). This extension to the MEM is particularly useful when estimated autocorrelation functions derived from long records of data have been produced by special-purpose digital processors. In this case a more efficient calculation of the maximum-entropy spectrum can be made then by using the method of Burg.

3. THE BURG MAXIMUM-ENTROPY METHOD*

The Burg maximum-entropy method of spectral analysis is particularly suited to the analysis of short data records when estimates of the autocorrelation function have not been given. For purposes of the derivation, however, it is easier at first to suppose that a series of M+1 equi-spaced complex autocorrelation function values $\phi(k)$, $k=0,1,\ldots,M$ are known. Here $\phi(k)$ is the autocorrelation with lag $k\Delta t$ where Δt is the sampling interval. From the usual definition of the autocorrelation function, $\phi(-k) = \phi^*(k)$, where the asterisk denotes complex conjugation.

The power spectrum P(f) estimated by the maximum-entropy method is given by

$$P(f) = \frac{P_M \Delta t}{\left| 1 - \sum_{k=1}^{M} a_{Mk} \exp[-j2\pi\, fk\, \Delta t] \right|^2} \tag{1}$$

* *The exposition which follows is an extension of the work of Andersen (1974), except that here the data are assumed to be complex rather than real. The present results are identical to those obtained by the more rigorous derivation by Smylie et al. (1973).*

where the frequency f is limited to the Nyquist interval, $-1/(2\Delta t) \leq f \leq 1/(2\Delta t)$, and the parameter P_M and the coefficients a_{Mk} are determined by solving the matrix equation

$$\begin{bmatrix} \phi(0) & \phi^*(1) & \cdots & \phi^*(M) \\ \phi(1) & \phi(0) & & \\ \vdots & & & \phi^*(1) \\ \phi(M) & \cdots\cdots & \phi(1) & \phi(0) \end{bmatrix} \begin{bmatrix} 1 \\ -a_{M1} \\ \vdots \\ -a_{MM} \end{bmatrix} = \begin{bmatrix} P_M \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2}$$

Equations (2) show that P_M is the output power of the (M+1)-length prediction-error filter $(1, -a_{M1},\ldots,-a_{MM})$. An (M+1)-length prediction-error filter gives as its output the difference between the k^{th} datum and a linear estimate of its value based on the preceeding M data.

The concept that P(f) as given by (1) is a spectral estimator can be understood as follows. Equation (2) shows that the output of the (M+1)-length prediction-error filter is a single pulse, which has a white power spectrum $P_M\Delta t$. This prediction-error filter is thus a filter which whitens the power spectrum of the input process. The reciprocal of the power spectrum of the whitening filter therefore gives an estimate of the power spectrum of the process.

In general, after the solution of (2) for order M has been obtained, it is desired to extrapolate to order M+1. The solution for order M+1 involves the determination of a set of M+3 unknowns $(a_{M+1,1},\ldots,a_{M+1,M+1};\ \phi(M+1);P_{M+1})$ from the M+2 eqns. (2). Thus an additional criterion is required in order to find a unique solution. Burg (1968) suggested that this criterion be based on that choice of $a_{M+1,M+1}$ which minimizes the output power P_{M+1}, where P_{M+1} is the average of the output power of the prediction error filter operating on the data in both the forward and reverse directions.

For the trivial case M=0, P_o is estimated by

$$P_o = \frac{1}{N} \sum_{t=1}^{N} |x_t|^2 \tag{3}$$

where P_o is an estimate of $\phi(0)$ and N is the number of complex data samples x_t. It is again assumed that the sampling interval is Δt.

For the case M=1, the two-point prediction-error filter $(1, -a_{11})$ is found by minimizing the average output power Π_1 with respect to a_{11}, where Π_1 is given by

$$\Pi_1 = \frac{1}{2}\frac{1}{N-1} \sum_{t=1}^{N-1} \left\{ |x_{t+1} - a_{11}\, x_t|^2 + |x_t - a_{11}^*\, x_{t+1}|^2 \right\} \tag{4}$$

The first term in the summation is the forward-direction output power of the filter and the second is the reverse-direction output power. The use of the complex conjugate of the filter coefficients is a consequence of the requirement that the power spectra of the forward and reverse direction filters be identical. If the data were real, then $x_t^* = x_t$, and as a consequence the a's would be real as well ($a_{11}^* = a_{11}$).

Maximizing Π_1 as a function of a_{11} yields

$$a_{11} = \frac{2 \sum_{t=1}^{N-1} x_t^* x_{t+1}}{\sum_{t=1}^{N-1} \left\{ |x_t|^2 + |x_{t+1}|^2 \right\}} . \qquad (5)$$

This step completes the first part of the inductive proof of a recursive procedure for obtaining the sets of a's and P's from a given set of complex data. Substitution of the value for a_{11} from (5) into eqn. (4) yields, $\Pi_{1,min} = P_1$, which may then be used to make an estimate of $\phi(1)$ from eqns. (2). In order to evaluate the a's, however, explicit estimates of the ϕ's are not required. This fact will become evident below.

A useful set of recursion relations between the filter coefficients $a_{M-1,k}$ and $a_{M,k}$ and the output powers P_{M-1} and P_M will now be derived, following Burg (1968). From eqn. (2) can be written recursive matrix relations to link the $(M-1)^{th}$-order filter to the M^{th}-order filter, when M+1 values of $\phi(k)$ are known:

$$\begin{bmatrix} \phi(0) & \cdots & \phi^*(M) \\ \vdots & \ddots & \vdots \\ \phi(M) & \cdots & \phi(0) \end{bmatrix} \begin{bmatrix} 1 \\ -a_{M1} \\ \vdots \\ -a_{M,M-1} \\ -a_{MM} \end{bmatrix} =$$

$$\begin{bmatrix} \phi(0) & \cdots & \phi^*(M) \\ \vdots & \ddots & \vdots \\ \phi(M) & \cdots & \phi(0) \end{bmatrix} \left\{ \begin{bmatrix} 1 \\ -a_{M-1,1} \\ \vdots \\ -a_{M-1,M-1} \\ 0 \end{bmatrix} - a_{MM} \begin{bmatrix} 0 \\ -a^*_{M-1,M-1} \\ \vdots \\ -a^*_{M-1,1} \\ 1 \end{bmatrix} \right\} \qquad (6)$$

and

$$\begin{bmatrix} P_{M-1} \\ 0 \\ \vdots \\ 0 \\ \Delta_{M-1} \end{bmatrix} - a_{MM} \begin{bmatrix} \Delta^*_{M-1} \\ 0 \\ \vdots \\ 0 \\ P_{M-1} \end{bmatrix} = \begin{bmatrix} P_M \\ 0 \\ \vdots \\ \\ 0 \end{bmatrix} \tag{7}$$

where

$$\Delta_{M-1} = \phi(M) - \sum_{k=1}^{M-1} a_{M-1,M-k}\, \phi(k) \tag{8}$$

Equations (6) and (7) lead immediately to the recursion equations

$$a_{M,k} = a_{M=1,k} - a_{MM} a^*_{M-1,M-k}, \quad 1 \le k \le M-1 \tag{9}$$

and

$$P_M = (1 - |a_{MM}|^2) P_{M-1} \tag{10}$$

Additional useful definitions are

$$a_{MO} = -1 \tag{11}$$

and

$$a_{Mk} = 0 \quad (k \ge M) \tag{12}$$

From eqn. (9), the average output power Π_M of the M^{th} order filter is then

$$\Pi_M = \frac{1}{2(N-M)} \sum_{t=1}^{N-M} \left(\left| \sum_{k=0}^{M} a^*_{Mk}\, x_{t+k} \right|^2 + \left| \sum_{k=0}^{M} a_{Mk}\, x_{t+M-k} \right|^2 \right)$$

$$= \frac{1}{2(N-M)} \sum_{t=1}^{N-M} \left(\left| \sum_{k=0}^{M} a^*_{M-1,k}\, x_{t+k} - a^*_{MM} \sum_{k=0}^{M} a_{M-1,M-k}\, x_{t+k} \right|^2 \right.$$

$$\left. + \left| \sum_{k=0}^{M} a_{M-1,k}\, x_{t+M-k} - a_{MM} \sum_{k=0}^{M} a^*_{M-1,M-k}\, x_{t+M-k} \right|^2 \right)$$

or

$$\Pi_M = \frac{1}{2(N-M)} \sum_{t=1}^{N-M} \left(|b_{Mt} - a^*_{MM} b'_{Mt}|^2 + |b'_{Mt} - a_{MM} b_{Mt}|^2 \right) \tag{13}$$

where

$$b_{Mt} = \sum_{k=0}^{M} a^*_{M-1,k}\, x_{t+k} = \sum_{k=0}^{M} a^*_{M-1,M-k}\, x_{t+M-k} \tag{14a}$$

$$b'_{Mt} = \sum_{k=0}^{M} a_{M-1,k}\, x_{t+M-k} = \sum_{k=0}^{M} a_{M-1,M-k}\, x_{t+k} \tag{14b}$$

$$t = 1,2,\ldots,N-M$$

Since b_{Mt} and b'_{Mt} are independent of a_{MM}, the condition

$$\frac{\partial}{\partial a_{MM}} \Pi_M = 0 \tag{15}$$

gives

$$a_{MM} = 2 \sum_{t=1}^{N-M} b^*_{Mt} b'_{Mt} \Bigg/ \sum_{t=1}^{N-M} \left(|b_{Mt}|^2 + |b'_{Mt}|^2 \right) \tag{16}$$

Furthermore

$$\frac{\partial}{\partial a^2_{MM}} \Pi_M = \frac{1}{N-M} \sum_{t=1}^{N-M} \left(|b_{Mt}|^2 + |b'_{Mt}|^2 \right) > 0 \tag{17}$$

so that the extremum of Π_M for a_{MM} given by eqns. (14a) and (14b) is a minimum.

Useful recursion formulas for the arrays b_{Mt} and b'_{Mt} are derived by means of (9) and (14a) and (14b):

$$b_{Mt} = b_{M-1,t} - a^*_{M-1,M-1}\, b'_{M-1,t} \tag{18a}$$

$$b'_{Mt} = b'_{M-1,t+1} - a_{M-1,M-1}\, b_{M-1,t+1} \tag{18b}$$

This completes the second part of the inductive proof. It has now been shown that the filter for any order M can be constructed from the filter of order M-1.

It can be seen that the arrays b_{Mt} and b'_{Mt} are constructed from the arrays $b_{M-1,t}$ and $b'_{M-1,t}$ by a simple linear operation. The starting values are

$$b_{0t} = b'_{0t} = x_t \tag{19}$$

but in practice the iterative procedure is stated at M=1 and the following values are used instead:

$$b_{1t} = x_t \tag{20a}$$

$$b'_{1t} = x_{t+1} \tag{20b}$$

$$t = 1,2,\ldots,N-1.$$

From eqn. (16) it is apparent that $|a_{MM}| \leq 1$; therefore, from eqn. (9), it follows that $0 \leq P_M \leq P_{M-1}$.

The inverse of the autocorrelation matrix ϕ can be expressed in terms of the filter coefficients and the output powers (Burg, 1968). If $\psi = \phi^{-1}$, then

$$\begin{aligned} \psi_{ij} &= \psi^*_{ji} = \psi_{M-i,M-j} = \psi^*_{M-j,M-i} \\ &= \sum_{k=0}^{n} a_{M-k,i-k} \cdot a^*_{M-k,j-k}/P_{M-k} \end{aligned} \tag{21}$$

$$i,j = 0,1,\ldots,M,$$

where n is the smaller of the two indices i and j. This inverse may be required in certain applications.

A criterion for the choice of the appropriate filter length M(<N) has been described by Akaike (1969a, b; 1970) and summarized by Ulrych and Bishop (1975). The criterion is known as teh Akaike final prediction error (FPE) and is defined as the mean-square prediction error. Numerically, the FPE of an M^{th} order filter is given by

$$(FPE)_M = \frac{N + (M+1)}{N - (M+1)} P_M \tag{22}$$

if the sample mean has been eliminated from the data, or

$$(FPE)_M = \frac{N + M}{N - M} P_M \tag{23}$$

otherwise*. The appropriate choice of M is then that which yields the minimum FPE. There appears to be no easy way of determining this minimum short of direct evaluation for each value of M, and care must be taken to avoid local minima induced by statistical fluctuations. Most authors suggest that a restriction of $M \leq N/2$ be observed to prevent the use of excessively long filters, which tend to artificially split what ought to be single spectral peaks (Chen and Stegen, 1974).†

A final observation (Andersen, 1974) is that eqn. (1) gives a detailed picture of the whole power spectrum, but if it is desired only to determine the frequency and magnitude of the principal peaks of the spectrum, it may be more appropriate to determine the positions of the complex roots of the polynomial

$$p(z) = 1 - \sum_{k=1}^{M} a_{MK} z^{-k} \qquad (24)$$

relative to the unit circle, and evaluate (1) only in those regions where $p(e^{j2\pi f\Delta t}) \approx 0$.

4. COMPUTING THE MAXIMUM ENTROPY SPECTRUM

Appendix A contains the listing of a Fortran subroutine CMESA, which computes the residual powers $P_1, P_2, \ldots, P_M$ and the filter coefficients a_{Mk}, $k = 1,2,\ldots,M$ from a given set of complex data.

Appendix B contains the listing of a Fortran subroutine CMESAP, which computes the magnitude of the power spectrum at a particular frequency according to (1).

The roots of the polynomial p(z) of eqn. (24) can be calculated using the subroutine CPOLY (Jenkins and Traub, 1972; Withers, 1974). A tested copy of this subroutine package is available from the author of this document.

5. CONCLUSION

The literature of the maximum-entropy method of spectral analysis has been briefly summarized, and the more significant references have been singled out as essential reading for the potential user.

The Burg method (Section 3) appears to offer a way of resolving spectral peaks in short data records when such peaks are separated by at least the order of the reciprocal of the length of the data record. Limited simulation experimentation by the author, using two complex sinusoidal signals

---

* *If the process being investigated has zero mean and can give rise to signals at frequencies near zero, then the sample mean should not be eliminated from the data and eqn. (23) is the appropriate formula to use.*

† *But see also Jones (1976).*

differing in frequency by about a half a cycle over the record length, has indicated that for such data the computed spectra were very sensitive to the relative phases of the sinusoids. In particular, good resolution was obtained when the sinusoids were in phase quadrature at the mid-point of the record, and spurious results were obtained in all other cases. This result apparently occurred because the cross-product terms in the short estimates of the auto-correlation functions did not cancel.

The development of the algorithm for the processing of the complex-valued data has been given in detail and a Fortran version of its implementation has been included. It is hoped that this Technical Note will assist those involved in the evaluation of the applicability of MEM spectral analysis to their particular sets of data.

6. ACKNOWLEDGEMENT

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A P P E N D I X   A

Subroutine CMESA

```
      SUBROUTINE CMESA(X,N,M,A,P,AA,B1,B2)
C
C     HERRING  21 APRIL, 1976.
C
C     SUBROUTINE FOR CALCULATING COMPLEX MAXIMUM ENTROPY
C          SPECTRAL ANALYSIS (CMESA) FILTER COEFFICIENTS
C          OF ORDER M AND RESIDUAL POWERS P FROM ORDER 1
C          TO ORDER M BASED ON THE N COMPLEX DATA X.
C
C     ARGUMENTS:
C
C          INPUT:
C
C               X      ARRAY OF COMPLEX DATA.
C               N      DIMENSION OF ARRAY X.
C               M      ORDER OF SET OF CMESA COEFFICIENTS
C                           TO BE COMPUTED.
C
C          OUTPUT:
C
C               A      COMPLEX ARRAY OF DIMENSION (M)
C                           CONTAINING CMESA FILTER
C                           COEFFICIENTS.
C               P      REAL ARRAY OF DIMENSION M CONTAINING
C                           RESIDUAL POWERS P(1),P(2),...,P(M)
C
C          WORKING STORAGE:
C
C               AA     COMPLEX ARRAY OF DIMENSION (M-1)
C                           OR GREATER.
C               B1,B2 COMPLEX ARRAYS OF DIMENSION (N-1)
C                           OR GREATER.
C
C     REFERENCE: ADAPTED FROM N. ANDERSEN, GEOPHYSICS,
C                VOL. 39, NO. 1 (FEBRUARY 1974)
C                PP. 69-72.
C
```

```
      COMPLEX X(N),A(M),AA(1),B1(1),B2(1),AMM,NUM
      INTEGER T
      REAL P(M)
C
      PLAST = 0.
      DO 110 T = 1,N
           PLAST = PLAST+REAL(X(T))*REAL(X(T))+AIMAG(X(T))*AIMAG(X(T))
110   CONTINUE
      PLAST = PLAST/FLOAT(N)
      MM = 1
      B1(1) = X(1)
      B2(N-1) = X(N)
      DO 120 T = 2,N-1
           B1(T) = B2(T-1) = X(T)
120   CONTINUE
      GO TO 300
C
200   MLAST = MM
      MM = MM+1
      DO 210 T = 1,MLAST
           AA(T) = A(T)
210   CONTINUE
      DO 220 T = 1,N-MM
           B1(T) = B1(T)-CONJG(AMM)*B2(T)
           B2(T) = B2(T+1)-AMM*B1(T+1)
220   CONTINUE
C
300   NUM = (0.,0.)
      DEN = 0.
      DO 310 T = 1,N-MM
           NUM = NUM+CONJG(B1(T))*B2(T)
           DEN = DEN+REAL(B1(T))**2+AIMAG(B1(T))**2
     1               +REAL(B2(T))**2+AIMAG(B2(T))**2
```

```
310   CONTINUE
      A(MM) = AMM = 2.*NUM/DEN
      P(MM) = PLAST*(1.-REAL(AMM)*REAL(AMM)-AIMAG(AMM)*AIMAG(AMM))
      PLAST = P(MM)
C
      IF (MM .EQ. 1)  GO TO 200
C
      DO 410 T = 1,MLAST
          A(T) = AA(T)-AMM*CONJG(AA(MM-T))
410   CONTINUE
C
      IF (MM .LT. M)  GO TO 200
C
      RETURN
      END
```

A P P E N D I X   B

Subroutine CMESAP

```
      SUBROUTINE CMESAP(M,A,PM,DELT,F,PF)
C
C     HERRING  22 APRIL, 1976.
C
C     SUBROUTINE FOR EVALUATING THE MAGNITUDE PF OF THE
C          MAXIMUM ENTROPY SPECTRUM AT ANY FREQUENCY F
C          USING A SET OF COMPLEX FILTER COEFFICIENTS A
C          GENERATED BY THE SUBROUTINE CMESA.
C
C     ARGUMENTS:
C
C          INPUT:
C
C               M      ORDER OF THE SET OF CMESA
C                      COEFFICIENTS TO BE USED.
C               A      COMPLEX ARRAY OF DIMENSION M CONTAINING
C                      THE CMESA FILTER COEFFICIENTS.
C               PM     THE REAL RESIDUAL POWER P(M) AS
C                      COMPUTED BY CMESA.
C               DELT   THE REAL TEMPORAL OR SPATIAL INCREMENT
C                      BETWEEN THE COMPLEX DATA SAMPLES
C                      INPUT TO CMESA.
C               F      THE REAL TEMPORAL OR SPATIAL FREQUENCY
C                      AT WHICH PF IS TO BE EVALUATED.
C
C          OUTPUT:
C
C               PF     THE DESIRED REAL ESTIMATE OF THE
C                      CMESA SPECTRUM AT THE FREQUENCY F.
C
      COMPLEX A(M),CSUM
      DATA TWOPI/6.2831853071795B6/
      OMEGA = -TWOPI*F*DELT
      CSUM = (0.,0.)
      DO 110 N = 1,M
           ARG = OMEGA*FLOAT(N)
           CSUM = CSUM+A(N)*CMPLX(COS(ARG),SIN(ARG))
  110 CONTINUE
      PF = PM*DELT/((1.-REAL(CSUM))**2+AIMAG(CSUM)*AIMAG(CSUM))
      RETURN
      END
```